A Note on Finitely Additive Strong Laws

by

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Suppose that X is a non-empty set with the discrete topology, $H = X^{\infty}$ with the product topology, $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ is an independent strategy on H, and $\{Y_n | n = 1, 2, \ldots\}$ is a sequence of coordinate mappings defined on H (cf. [2]). Consider the following six statements.

(a) For any $\varepsilon > 0$, $\sum_{K=1}^{\infty} \sigma([h| \frac{|S_K(h)|}{K} > \varepsilon]) < \infty$

(b) For any $\varepsilon > 0$, $\sum_{n=1}^{\infty} \sigma([h| \frac{|S_{2^n}(h)|}{2^n} > \varepsilon]) < \infty$

(c) For any $\varepsilon > 0$, $\sum_{n=1}^{\infty} \sigma([h| \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \varepsilon]) < \infty$

(d) $\sigma([h| \lim_{n\to\infty} \frac{S_{2^n}(h)}{2^n} = 0]) = 1$

(e) $\frac{S_n}{n}$ converges to 0 in σ-probability as $n \to \infty$

(f) $\sigma([h| \lim_{n\to\infty} \frac{S_n(h)}{n} = 0]) = 1$

Where, for each $n = 1, 2, 3 \ldots$, $S_n = \sum_{j=1}^{n} Y_j$. Then, we have the following results (Theorems 1-1, 1-2, 1-3, 1-4, and 1-5). Based on Theorems 1-1 and 1-2, we have another proof of a finitely additive version of the strong law of large numbers (see Theorem 4-1 in [2] and Theorem 1-6 below).

1. Main Theorems.

Theorem 1-1.

Statements (c) and (d) are equivalent.

Proof: (c) $\Rightarrow$ (d).

Let $\{\varepsilon_j | j = 1, 2, \ldots\}$ be a strictly decreasing sequence of positive real numbers such that $\lim_{j\to\infty} \varepsilon_j = 0$. Now, let $N_0 = 0$, and, for each $j = 1, 2, 3, \ldots$ let

$$N_j = \max\{N_{j-1}+1,\ \inf\{K | \sum_{n=K}^{\infty} \sigma([h| \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \varepsilon_j]) \le \frac{1}{j^{1+\alpha}}\}\},$$

where α is a positive constant. It is easy to check that,

(1) $N_j < N_{j+1}$, $N_j < \infty$ for all $j = 1, 2, 3\ldots$ and $\lim_{j\to\infty} N_j = \infty$.

(2) $\sum_{j=1}^{\infty} \sum_{n=N_j}^{\infty} \sigma([h| \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \varepsilon_j]) < \infty.$

Next, for each $j = 1, 2, 3\ldots$, each positive integer n, if

$$N_j \le n < N_{j+1}, \quad \text{let} \quad L_n = [h| \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \varepsilon_j] .$$

Then,

$$\sum_{n=N_1}^{\infty} \sigma(L_n) = \sum_{j=1}^{\infty} \sum_{N_j \le n < N_{j+1}} \sigma([h| \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \varepsilon_j])$$

$$\le \sum_{j=1}^{\infty} \sum_{n \ge N_j} \sigma([h| \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \varepsilon_j]) < \infty .$$

By Theorem 3-3 of [2], we have $\sigma([L_n \text{ i, } O(n)]) = 0$ which is equivalent to $\sigma([L_n \text{ i, } O(n)]^c) = 1$. Now, we show that

$$[L_n \text{ i, } O(n)]^c \subseteq [h| \lim_{n\to\infty} \frac{S_{2^n}(h)}{2^n} = 0].$$

Suppose that h is in $[L_n\ i,\ O(n)]^c$, it is sufficient to show that $\overline{\lim}_{n\to\infty} \frac{|S_{2^n}(h)|}{2^n} < \varepsilon$ for any $\varepsilon > 0$. Let $\varepsilon > 0$, first, we choose a positive integer J^* so large such that $\varepsilon_j \le \frac{\varepsilon}{2}$ if $j \ge J^*$ (since $\lim_{j\to\infty} \varepsilon_j = 0$). Since h is in $[L_n\ i,\ O(n)]^c$, there exists a positive integer J^{**} such that $h \in L_n^c$ if $n \ge N_{J^{**}}$. Now, let $J = \max(J^*, J^{**})$. Then, for all $i \ge J$, $\varepsilon_i \le \varepsilon_J \le \varepsilon_{J^*} \le \frac{\varepsilon}{2}$ (since $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots$) and

$$|S_{2^{n+1}}(h) - S_{2^n}(h)| \le 2^n\,\varepsilon_i \le 2^n(\tfrac{\varepsilon}{2}) \quad \text{if} \quad N_i \le n < N_{i+1}\ .$$

Now, for any $K \ge N_J + 1$, then

$$\frac{|S_{2^K}(h)|}{2^K} \le \frac{1}{2^K}\left(\sum_{\ell=N_J}^{K-1} |S_{2^{\ell+1}}(h) - S_{2^\ell}(h)| + |S_{2^{N_J}}(h)|\right)$$

$$\le \frac{1}{2^K}\sum_{\ell=N_J}^{K-1} 2^\ell(\tfrac{\varepsilon}{2}) + \frac{1}{2^K}|S_{2^{N_J}}(h)| < \frac{\varepsilon}{2} + \frac{|S_{2^{N_J}}(h)|}{2^K}\ .$$

If we choose a large enough positive integer K, then, for all $k \ge K$, $\frac{|S_{2^{N_J}}(h)|}{2^K} \le \varepsilon/2$. Hence, if $k \ge K$, $\frac{|S_{2^K}(h)|}{2^K} < \varepsilon$, i.e., $\overline{\lim}_{n\to\infty} \frac{|S_{2^n}(h)|}{2^n} < \varepsilon$.

Therefore, we have

$$[L_n\ i,\ O(n)]^c \subseteq [h \mid \lim_{n\to\infty} \frac{S_{2^n}(h)}{2^n} = 0]\ .$$

Since

$$\sigma([L_n\ i,\ O(n)]^c) = 1, \quad \sigma([h \mid \lim_{n\to\infty} \frac{S_{2^n}(h)}{2^n} = 0]) = 1\ .$$

The proof of "(c) ⇒ (d)", now is complete. (d) ⇒ (c) since

$$\sigma([h|\lim_{n\to\infty} \frac{S_{2^n}(h)}{2^n} = 0]) = 1, \quad \sigma([h|\lim_{n\to\infty} \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} = 0]) = 1.$$

Hence, for any $\epsilon > 0$, $\sigma([h|\frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \epsilon \quad i, O(n)]) = 0.$

By Theorem 3-3 of [2], we have, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \sigma([h|\frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \epsilon]) < \infty .$$

The proof of "(d) ⇒ (c)", now is complete.

Theorem 1-2.

Statements (c) and (e) imply Statement (f).

Proof: As in the proof of Theorem 1, let $\{\epsilon_j | j = 1, 2, 3,\ldots\}$ be a strictly decreasing sequence of positive real numbers such that $\lim_{j\to\infty} \epsilon_j = 0$. Let $N_0^* = 0$, and for each $j = 1, 2, 3,\ldots,$ let

$$N_j^* = \max\{N_{j-1}^* + 1, \inf\{K| \sum_{n=K}^{\infty} \sigma([h|\frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \epsilon_j]) \le \frac{1}{j^{1+\alpha}}\}\} ,$$

where α is a positive constant. It is easy to check that $N_j^* < \infty$, $N_j^* < N_{j+1}^*$ for all $j = 1, 2, \ldots,$ $\lim_{j\to\infty} N_j^* = \infty$, and

$$\sum_{j=1}^{\infty} \sum_{n=N_j^*}^{\infty} \sigma([h|\frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \epsilon_j]) < \infty .$$

By Statement (e), we have, for each $j = 1, 2, 3\ldots,$ there exists a positive integer N_j^{**} such that if

$$K \geq 2^{N_j^{**}}, \quad \sigma([h| \frac{|S_K(h)|}{K} > \frac{\epsilon_j}{4}]) \leq \frac{1}{4} .$$

We can and do assume that $N_1^{**} < N_2^{**} < \ldots$ and $\lim_{j\to\infty} N_j^{**} = \infty$. Now, for each $j = 1, 2, 3, \ldots,$ let $N_j = \max(N_j^*, N_j^{**})$. Then, for each $j = 1, 2, 3, \ldots,$

$$\sigma([h| \frac{|S_K(h)|}{K} > \frac{\epsilon_j}{4}]) \leq \frac{1}{4} \text{ if } K \geq 2^{N_j}, \text{ and}$$

$$\sum_{j=1}^{\infty} \sum_{n=N_j}^{\infty} \sigma([h| \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \epsilon_j]) < \infty .$$

Next, if $N_j \leq n < N_{j+1}$ $(j = 1, 2, 3, \ldots)$, $2^n \leq K < 2^{n+1}$, then

$$\sigma([h| \frac{|S_{2^{n+1}}(h) - S_K(h)|}{2^n} > \epsilon_j])$$

$$\leq \sigma([h| \frac{|S_{2^{n+1}}(h)|}{2^{n+1}} > \frac{\epsilon_j}{4}]) + \sigma([h| \frac{|S_K(h)|}{K} > \frac{\epsilon_j}{4}])$$

$$\leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} .$$

Hence, if $N_j \leq n < N_{j+1}$ $(j = 1, 2, \ldots)$, then

$$\max_{2^n \leq K \leq 2^{n+1}} \sigma([h| \frac{|S_{2^{n+1}}(h) - S_K(h)|}{2^n} > \epsilon_j]) \leq \frac{1}{2} .$$

By Lemma 1-3 of [1], we have

$$\sigma([h| \max_{2^n < K \leq 2^{n+1}} \frac{|S_K(h) - S_{2^n}(h)|}{2^n} > 2\,\epsilon_j])$$

$$\leq \frac{1}{1-\frac{1}{2}}\,\sigma([h|\frac{|s_{2^{n+1}}(h) - s_{2^n}(h)|}{2^n} > \epsilon_j])$$

$$= 2\,\sigma([h|\frac{|s_{2^{n+1}}(h) - s_{2^n}(h)|}{2^n} > \epsilon_j]) ,$$

for each $j = 1, 2, 3, \ldots,$ and each positive integer n such that $N_j \leq n < N_{j+1}$. Now, for each $j = 1, 2, 3, \ldots,$ each positive integer n such that $N_j \leq n < N_{j+1}$, let

$$L_n = [h| \max_{2^n < \ell \leq 2^{n+1}} \frac{|s_\ell(h) - s_{2^n}(h)|}{2^n} > 2\,\epsilon_j] .$$

Then,

$$\sigma(L_n) \leq 2\,\sigma([h|\frac{|s_{2^{n+1}}(h) - s_{2^n}(h)|}{2^n} > \epsilon_j]) ,$$

if $N_j \leq n < N_{j+1}$ $(j = 1, 2, 3, \ldots)$. Therefore,

$$\sum_{n=N_1}^{\infty} \sigma(L_n) = \sum_{j=1}^{\infty} \sum_{N_j \leq n < N_{j+1}} \sigma(L_n)$$

$$\leq 2 \sum_{j=1}^{\infty} \sum_{N_j \leq n < N_{j+1}} \sigma([h|\frac{|s_{2^{n+1}}(h) - s_{2^n}(h)|}{2^n} > \epsilon_j])$$

$$\leq 2 \sum_{j=1}^{\infty} \sum_{n \geq N_j} \sigma([h|\frac{|s_{2^{n+1}}(h) - s_{2^n}(h)|}{2^n} > \epsilon_j]) < \infty .$$

By Theorem 3-3 of [2], we have $\sigma([L_n\ i,\ 0(n)]^c) = 1$.

By Theorem 1-1, Statement (c) implies that

$$\sigma([h| \lim_{n\to\infty} \frac{s_{2^n}(h)}{2^n} = 0]) = 1.$$

Now, notice that

$$[L_n\ i,\ O(n)]^c \cap [h|_{n \to \infty} \lim \frac{S_{2^n}(h)}{2^n} = 0] \subseteq [h|\ \lim_{n \to \infty} \frac{S_n(h)}{n} = 0] .$$

Therefore,

$$\sigma([h|\ \lim_{n \to \infty} \frac{S_n(h)}{n} = 0]) = 1$$

and the proof of Theorem 1-2, now, is complete.

Corollary 1-1.

Statements (d) and (e) imply Statement (f).

Proof: By Theorem 1-1, we have Statement (c), and by Theorem 1-2, we get Corollary 1-1.

In [3], Purves and Sudderth showed that "suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H, then σ is countably additive when restricted to the σ-field of tail sets in $a(\sigma)$". Using this result, there are other proofs available for Theorem 1-1 and Theorem 1-2.

Theorem 1-3.

Statement (b) implies Statements (c) and (d).

Proof: Notice that, for each $n = 1, 2, 3, \ldots,$ each $\epsilon > 0$, the set

$[h|\ \ |S_{2^{n+1}}(h) - S_{2^n}(h)| > 2^n\ \epsilon]$ is contained by the set

$$[h|\ \ |S_{2^{n+1}}(h)| > 2^{n+1} \cdot \frac{\epsilon}{4}] \cup [h|\ \ |S_{2^n}(h)| > 2^n \cdot \frac{\epsilon}{4}] .$$

Therefore, for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \sigma([h|\frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \epsilon])$$

$$\leq \sum_{n=1}^{\infty} \sigma([h|\frac{|S_{2^{n+1}}(h)|}{2^{n+1}} > \frac{\epsilon}{4}]) + \sum_{n=1}^{\infty} \sigma([h|\frac{|S_{2^n}(h)|}{2^n} > \epsilon]) < \infty .$$

By Theorem 1-1 and above result, we also, have the implication of "(b) ⇒ (d)".

Corollary 1-2.

Statements (b) and (e) imply Statement (f).

Proof: By Theorem 1-3, we have Statement (c) and by Theorem 1-2, we get Corollary 1-2.

Theorem 1-4.

Statement (a) implies Statements (b), (c),(d), (e), and (f).

Proof: The implications of "(a) ⇒ (b), (a) ⇒ (c), (a) ⇒ (d), and, (a) ⇒ (e)" are obvious. By Theorem 1-2, we have the implication of "(a) ⇒ (f)".

Theorem 1-5.

Statement (f) implies Statements (c) and (d).

Proof: The implication of "(f) ⇒ (d)" is obvious and the implication of "(f) ⇒ (c)" is implied by Theorem 1-1 and the implication of "(f) ⇒ (d)".

Theorem 1-6. (A finitely additive version of the strong law of large numbers.)

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H and $\{Y_n | n = 1, 2, \ldots\}$ is a sequence of coordinate mappings defined on H.

Suppose that $\sigma(Y_n) = 0$, $\sigma(|Y_n|^{2r}) < \infty$ for all $n = 1, 2, \ldots,$ where r is a positive real number and $r \geq 1$. Then, if

$$\sum_{n=1}^{\infty} \frac{\sigma(|Y_n|^{2r})}{n^{1+r}} < \infty, \quad \sigma([h| \lim_{n\to\infty} \frac{S_n(h)}{n} = 0]) = 1.$$

Proof:

By Kronecker's Lemma, we have

$$\lim_{n\to\infty} \frac{1}{n^{1+r}} \{\sum_{j=1}^{n} \sigma(|Y_j|^{2r})\} = 0.$$

Hence, for any $\epsilon > 0$,

$$\sigma([h| \frac{|S_n(h)|}{n} > \epsilon])$$

$$\leq \frac{1}{\epsilon^{2r}} \times \frac{\sigma(|S_n|^{2r})}{n^{2r}} \leq \frac{1}{\epsilon^{2r}} \cdot \frac{A\sum_{j=1}^{n} \sigma(|Y_j|^{2r})}{n^{1+r}}$$, where A is a positive constant.

(The last step is implied by Lemma 2-4 in [2]).

Therefore, $\lim_{n\to\infty} \sigma([h| \frac{|S_n(h)|}{n} > \epsilon]) = 0$ for any $\epsilon > 0$, i.e., $\frac{S_n}{n} \overset{\sigma}{\to} 0$ as $n \to \infty$. For any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \sigma([h| \frac{|S_{2^{n+1}}(h) - S_{2^n}(h)|}{2^n} > \epsilon])$$

$$\leq \sum_{n=1}^{\infty} \frac{\sigma(|S_{2^{n+1}} - S_{2^n}|^{2r})}{\epsilon^{2r}(2^n)^{2r}} \leq \sum_{n=1}^{\infty} \frac{A\,(2^n)^{r-1} \sum_{j=2^n+1}^{2^{n+1}} \sigma(|Y_j|^{2r})}{\epsilon^{2r}(2^n)^{2r}}$$

$$\leq \frac{2^{1+r}}{\epsilon^{2r}} \sum_{n=1}^{\infty} \frac{A}{(2^{n+1})^{1+r}} \sum_{j=2^n+1}^{2^{n+1}} \sigma(|Y_j|^{2r}) \leq \frac{2^{1+r}}{\epsilon^{2r}} A \sum_{j=1}^{\infty} \frac{\sigma(|Y_j|^{2r})}{j^{1+\gamma}} < \infty .$$

Therefore, by Theorem 1-2, we have

$$\sigma([h|\lim_{n\to\infty} \frac{S_n(h)}{n} = 0]) = 1 .$$

2. Remarks.

1. Statements (c) and (d) can be replaced by the following general forms "(c'), for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \sigma([h|\frac{|S_{N_{j+1}}(h) - S_{N_j}(h)|}{N_j} > \epsilon]) < \infty.$$

(d')

$$\sigma([h|\lim_{n\to\infty} \frac{S_{N_j}(h)}{N_j} = 0]) = 1".$$ Where $\{N_j | j=1, 2,\ldots\}$ is a strictly increasing sequence of positive integers such that

$$\varliminf_{j\to\infty} \frac{N_{j+1}}{N_j} = a, \quad \varlimsup_{j\to\infty} \frac{N_{j+1}}{N_j} = b \quad \text{and} \quad 1 < a \le b < \infty .$$

2. In the conventional theory of probability, Statement (f) trivially implies Statement (e). But, it is false for our finitely additive setting. (See Example 1 below).

3. If, we have the following additional condition "there exists a strictly increasing sequence $\{N_j | j = 1, 2,\ldots\}$ of positive integers such that $\varlimsup_{j\to\infty} \frac{N_{j+1}}{N_j} = b < \infty$ and $\frac{1}{N_j} S_{N_j} \to 0$ in σ-probability as $j \to \infty$" too. Then, Statement (f) implies Statement (e).

Example 1.

Let $X = \{1, 2, 3, \ldots\}$, $H = X^{\infty}$, and $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ be an independent strategy on H such that γ_1 is a finitely additive probability measure defined on the class of all subsets of X and $\gamma_1(A) = 0$ if A is a finite subset of X. $\gamma_2, \gamma_3, \ldots$ are any probability measures defined on the class of all subsets of X. Let $Y_1, Y_2, \ldots$ be real-valued function defined on H by $Y_1(h) = x_1$ if $h = (x_1, x_2, \ldots) \in H$, and $Y_n(h) = 0$ for all $n = 2, 3, 4, \ldots$, all $h \in H$. It is easy to check that

$$\sigma([h|\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = 0]) = 1 .$$

But, $\frac{1}{n} \sum_{j=1}^{n} Y_j(h)$ does not converge to 0 in σ-probability as $n \to \infty$. This example also shows that none of Statements (c), (d), and (f) imply Statements (a) or (b).

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References

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